arXiv:math-ph/0404014v4 25 Jul 2004

Solutions for the General, Confluent and Biconfluent Heun equations and their connection with Abel equations

E.S. Cheb-Terrab[a,b]

[a] *CECM, Department of Mathematics*
Simon Fraser University, Vancouver, British Columbia, Canada.

[b] *Maplesoft, Waterloo Maple Inc.*

Abstract

In a recent paper, the canonical forms of a new multi-parameter class of Abel differential equations, so-called AIR, all of whose members can be mapped into Riccati equations, were shown to be related to the differential equations for the hypergeometric ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ and ${}_0\mathrm{F}_1$ functions. In this paper, a connection between the AIR canonical forms and the Heun General (GHE), Confluent (CHE) and Biconfluent (BHE) equations is presented. This connection fixes the value of one of the Heun parameters, expresses another one in terms of those remaining, and provides closed form solutions in terms of ${}_p\mathrm{F}_q$ functions for the resulting GHE, CHE and BHE, respectively depending on four, three and two irreducible parameters. This connection also turns evident what is the relation between the Heun parameters such that the solutions admit Liouvillian form, and suggests a mechanism for relating linear equations with N and N-1 singularities through the canonical forms of a non-linear equation of one order less.

Introduction

The Heun equation [1] is a second order linear equation of the form

$$y'' + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right) y' + \frac{\alpha\,\beta\,x - q}{x\,(x-1)\,(x-a)}\,y = 0, \tag{1}$$

where $\{\alpha, \beta, \gamma, \delta, \epsilon, a, q\}$ are constant with respect to x, are related by $\gamma+\delta+\epsilon = \alpha+\beta+1$, and $a \neq 0, a \neq 1$. This equation has four regular singular points, at $\{0, 1, a, \infty\}$. Through confluence processes, equation (1), herein called the General Heun Equation (GHE), transforms into four other multi-parameter equations [2], so-called Confluent (CHE), Biconfluent (BHE), Doubleconfluent (DHE) and Triconfluent (THE). Through transformations of the form $y \to P(x)\,y$, these five equations can be written in normal form[1], using the notation of [2], in terms of arbitrary constants $\{a,\, A, B, C, D, E, F\}$; for the 6-parameter GHE (1) we have

$$y'' + \left(\frac{A}{x} + \frac{B}{x-1} - \frac{A+B}{x-a} + \frac{D}{x^2} + \frac{E}{(x-1)^2} + \frac{F}{(x-a)^2}\right) y = 0 \tag{2}$$

The 5-parameter CHE,

$$y'' + \left(A + \frac{B}{x} + \frac{C}{x-1} + \frac{D}{x^2} + \frac{E}{(x-1)^2}\right) y = 0, \tag{3}$$

has two regular singularities at $\{0, 1\}$ and one irregular singularity at ∞. The 4-parameter BHE,

[1] A second order linear ODE is in *normal form* when the coefficient of y' is equal to zero - see the Appendix.

$$y'' + \left(-x^2 + B\,x + C + \frac{D}{x} + \frac{E}{x^2}\right) y = 0, \tag{4}$$

has one regular singularity at 0 and one irregular singularity at ∞. The 4-parameter DHE,

$$y'' + \left(A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3} + \frac{A}{x^4}\right) y = 0, \tag{5}$$

has two irregular singularities at $\{0, \infty\}$. The 3-parameter THE, with one irregular singularity at ∞, is

$$y'' + \left(-\frac{9}{4}\,x^4 + C\,x^2 + D\,x + E\right) y = 0 \tag{6}$$

Eq.(1), originally studied by Heun as a generalization of Gauss' hypergeometric (${}_p\mathrm{F}_q$) equation, as well as these related confluent families represented by (3–6), appear in applications in varied areas[2]. As a sample of recent related works, in [3] quase-normal modes of near extremal black branes are found solving a singular boundary value problem for (1); in [4], hyper-spherical harmonics, with applications in three-body systems, are developed in connection with the solutions of (1); in [5], a method of calculation of propagators for the case of a massive spin 3/2 field, for arbitrary space-time dimensions and mass, is developed in terms of the solutions of (1); in [6], parametric resonance after inflation is discussed in connection with the solutions of a particular form of (1). The separation of variables for the Schrödinger equation in a large number of problems results in Heun type equations too, typically for the radial coordinate, and also non-linear formulations involving Painlevé type equations [7] can be derived from Heun equations regarded as quantum Hamiltonians. A number of traditional equations of mathematical physics, as for instance the Lamé, spheroidal wave, and Mathieu equations, are also particular cases of Heun equations.

The solutions for these five Heun equations are the subject of current study [8]–[15]. In this paper, a hitherto unknown connection between Heun equations and a single multiparameter Abel equation [16], known to have canonical forms solvable in terms of ${}_p\mathrm{F}_q$ functions [17], is shown. This connection fixes one of the parameters and expresses another one in terms of those remaining in the Heun equations (2), (3) and (4), and provides exact closed form solutions for the resulting non-trivial 4-parameter GHE, 3-parameter CHE, and 2-parameter BHE families. The solutions are linear combinations involving ${}_2\mathrm{F}_1$ or ${}_1\mathrm{F}_1$ functions, and this connection with Abel equations also turns evident what is the relation between the Heun parameters such that the solutions of these three families admit Liouvillian form. From these results, an alternative approach to finding the same solutions, by exploring non-local transformations, is derived, and some of these solutions are shown to match those derived in [15] using an essentially different approach.

The multiparameter BHE, CHE and GHE equations solved in this paper are non-trivial in that they are irreducible, not degenerate, cases: the number of their singularities cannot be reduced and the equations cannot be mapped into ${}_p\mathrm{F}_q$ equations through extended transformations of the form

$$x \rightarrow \frac{\alpha x^k + \beta}{\gamma x^k + \delta}, \quad y \rightarrow P(x)\, y \tag{7}$$

where $\{\alpha, \beta, \gamma, \delta, k\}$ are constants and $P(x)$ is an arbitrary Liouvillian function. Hence, the solutions being presented cannot be reduced to ${}_p\mathrm{F}_q$ solutions of the form

$$y = P(x)\, {}_p\mathrm{F}_q\ \left(..., ..., \frac{\alpha x^k + \beta}{\gamma x^k + \delta}\right). \tag{8}$$

and it is in this extended sense that, herein, we say the equations being solved are not of ${}_p\mathrm{F}_q$ type[3].

Apart from being a way to relate Heun $\leftrightarrow$ ${}_p\mathrm{F}_q$ equations, leading to solutions to the former ones, this connection Heun $\leftrightarrow$ Abel is important in itself: Abel equations also appear frequently in applications [19]–[22] and through this connection it is possible to study their properties by studying those of the related

[2]For a list of applications of Heun's equations compiled in 1995 see p.340 of [1].

[3]There exist symbolic computation libraries that systematically resolve the equivalence of linear ODEs under (7) - see [18].

linear equations. It is implicit in the existence of this Heun $\leftrightarrow$ Abel link that there exists an equivalent link, between linear equations of order n with N and $N-1$ singularities and related confluent cases, through single non-linear "Abel-like" equations of order $n-1$.

The paper is organized as follows. In sec. 1, some results of [16] and [17] are reviewed and a connection between Heun and Abel equations is made explicit. In sec. 2, 3 and 4, the restrictions that this connection implies on the parameters entering the BHE, CHE and GHE equations are derived, and it is shown how a sequence mapping Heun $\rightarrow$ Abel $\rightarrow {}_p\mathrm{F}_q$ equations can be composed to obtain transformations mapping Heun $\rightarrow {}_p\mathrm{F}_q$ equations. When the aforementioned restriction on the Heun parameters holds, these transformations lead to closed form solutions for the GHE (2), CHE (3) and BHE (4), expressed in terms of exponentials of integrals of ${}_2\mathrm{F}_1$ or ${}_1\mathrm{F}_1$ functions. In sec. 5, taking advantage of the results of the previous sections, an alternative derivation is developed leading to solutions free of integrals. A discussion around these results is found in sec. 6 and 7. In an Appendix, the formulas relating the normal and canonical forms for the Heun equations are included for completeness, as well as symbolic computation input permitting the verification of the solutions presented.

1 A connection between Heun, Abel and ${}_p\mathrm{F}_q$ differential equations

The transformations being presented, relating Heun and ${}_p\mathrm{F}_q$ hypergeometric equations, were obtained by composing transformations which map Heun, Riccati, Abel and ${}_p\mathrm{F}_q$ equations according to the sequence

$$Heun \rightarrow Riccati \rightarrow Abel \rightarrow Abel_{canonical} \rightarrow Riccati \rightarrow {}_p\mathrm{F}_q$$

As we shall see in sec. 5, knowing the form of these transformations, one can re-derive them in an alternative way, shortcutting the step which goes through Abel equations. In this section, however, the *Heun* $\leftrightarrow$ *Abel* connection is kept visible: Abel equations are relevant by themselves and it was through this connection that the *Heun* $\leftrightarrow {}_p\mathrm{F}_q$ relation being presented became evident.

Abel equations of the second kind [20] are equations of the form

$$y' = \frac{f_3(x)\, y^3 + f_2(x)\, y^2 + f_1(x)\, y + f_0(x)}{g_1(x)\, y + g_0(x)}, \tag{9}$$

where the $\{f_i,\ g_i\}$ are arbitrary functions and either $f_3(x) \neq 0$ or $g_1(x) \neq 0$. In [16] it is shown that, departing from a Riccati type equation,

$$y' = h_2(x)\, y^2 + h_1(x)\, y + h_0(x), \tag{10}$$

by suitably restricting the form of the mappings h_i and making use of the *inverse* transformation $\{x \leftrightarrow y\}$[4], one can construct an Abel equation,

$$y' = \frac{(y-\rho_1)(y-\rho_2)(y-\rho_3)}{(s_2\, x^2 + s_1\, x + s_0)\, y + r_2\, x^2 + r_1\, x + r_0} \tag{11}$$

where the $\{s_i, r_i, \rho_i\}$ are constants, and $s_2 \neq 0$ or $r_2 \neq 0$. This Abel equation is representative of a multi-parameter class all of whose members can be transformed into Riccati equations (10) using $\{x \leftrightarrow y\}$, and from there into second order linear equations using the *Riccati* $\rightarrow$ *linear* mapping [20]

$$y \rightarrow -\frac{y'}{h_2(x)\, y} \tag{12}$$

The equations of this Abel class, named "Abel Inverse Riccati" (AIR) in [16], are then generated from (11) by applying to it class transformations of the form

$$\{x \rightarrow F(x), \quad y \rightarrow \frac{P_1(x)\, y + Q_1(x)}{P_2(x)\, y + Q_2(x)}\}, \tag{13}$$

[4]By $\{x \leftrightarrow y\}$ we mean changing variables $\{x = u(t),\ y(x) = t\}$ followed by renaming $\{u = y,\ t = x\}$.

where $\{F, P_1, P_2, Q_1, Q_2\}$ are arbitrary mappings with $F' \neq 0$, $P_1 Q_2 - P_2 Q_1 \neq 0$. The relevance of the AIR class can be inferred from the fact that most of the Abel solvable equations found in the literature[5] are shown in [16] to be particular members of AIR.

An important property of (11) is that its connection with second order linear equations, that is, its "Inverse Riccati" character, is invariant under Möbius (linear fractional) changes of x and y. This property is used in [17] to accomplish a full classification of (11) in terms of six canonical forms. For that purpose, through Möbius changes of y, (11) is first transformed into

$$y' = \frac{P(y)}{(s_2\, x^2 + s_1\, x + s_0)\, y + r_2\, x^2 + r_1\, x + r_0} \tag{14}$$

for some new constants $\{s_i\, r_i\}$, with $P(y)$ equal to $y\,(y-1)$, y or 1, respectively according to whether in (11) there are three, two or only one distinct roots ρ_i. As shown in [17], each of these three cases splits further into two subcases, and the six resulting canonical forms are solvable in terms of ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ and ${}_0\mathrm{F}_1$ functions; in this way, closed form ${}_p\mathrm{F}_q$ solutions can be constructed for the whole AIR class.

The key observation now is that the AIR equation (14) is also connected in a surprisingly simple manner to the Heun family of equations. As we shall see, by applying to (14) the transformation $\{x \leftrightarrow y\}$, one obtains a Riccati equation, and by transforming it further into a second order linear equation using (12), one directly obtains the GHE, CHE or BHE Heun equations (with some restrictions on the parameters), respectively according to the three possible values of $P(y)$. Since the AIR (14) admits solutions expressible using ${}_p\mathrm{F}_q$ functions for the three possible values of $P(y)$, the GHE, CHE and BHE Heun families which can respectively be derived from (14) also admit closed form solutions expressible in terms of these ${}_p\mathrm{F}_q$ functions.

2 Closed form solutions for a subfamily of the BHE

Considering first the simplest case, where the three roots ρ_i in (11) are equal, in (14) we have $P(y) = 1$ and so the AIR equation becomes

$$y' = \frac{1}{(s_2\, x^2 + s_1\, x + s_0)\, y + r_2\, x^2 + r_1\, x + r_0} \tag{15}$$

Recalling that either $s_2 \neq 0$ or $r_2 \neq 0$, changing variables using $\{x \leftrightarrow y\}$ we obtain the Riccati form

$$y' = (s_2\, x + r_2)\; y^2 + (s_1\, x + r_1)\; y + s_0\, x + r_0 \tag{16}$$

Using (12), this equation is transformed into the second order linear equation

$$y'' = \frac{\left(s_2\, s_1 x^2 + (s_2 r_1 + s_1 r_2)\, x + s_2 + r_2 r_1\right)}{s_2\, x + r_2}\, y' - \left(s_2 s_0 x^2 + (s_2 r_0 + s_0 r_2)\, x + r_2 r_0\right) y \tag{17}$$

This equation has one regular singularity at $-r_2/s_2$, one irregular singularity at ∞, and, by rewriting it in normal form, it is straightforward to verify that it is the BHE equation (4) with one of its four parameters fixed and two other ones interrelated. For that purpose, we note first that the case $s_2 = 0$ presents no interest since it directly simplifies (17) to a ${}_p\mathrm{F}_q$ equation. Assuming $s_2 \neq 0$ in (14), we take $s_2 = 1$ without loss of generality. To have the regular singularity of (17) located at 0, it suffices to take $r_2 = 0$, and, taking $s_0 = s_1^2/4 - 1$, the coefficient of x^2 in the normal form of (17) will be as in (4). In summary, using $y \rightarrow \sqrt{x}\,\exp((x\,(s_1\, x + 2\, r_1)\,/4\,)\, y$ to rewrite (17) in normal form at $\{s_2 = 1, r_2 = 0, s_0 = s_1^2/4 - 1\}$, the equation becomes

$$y'' = \left(x^2 + \left(\frac{s_1 r_1}{2} - r_0\right) x + \frac{{r_1}^2}{4} + \frac{r_1}{2\,x} + \frac{3}{4\,x^2}\right) y \tag{18}$$

which is the BHE (4) at $\{B = r_0 - s_1 r_1/2,\ D = -r_1/2,\ C = -D^2,\ E = -3/4\}$.

[5] For a collection of these see [23].

The relevance of this result is in that, on the one hand, (18) is a non-trivial 2-parameter form of the BHE for which solutions are not known in general; on the other hand, as shown in [17], the Abel equation (15), from which (18) is derived, is solvable in terms of ${}_1\mathrm{F}_1$ and ${}_0\mathrm{F}_1$ (Kummer and Bessel) hypergeometric functions. Therefore, a closed form solution for the BHE (18) can also be expressed using these ${}_p\mathrm{F}_q$ functions.

It is interesting, when possible, to separate the Liouvillian from the Non-Liouvillian solutions of (18), since the former ones represent the "special cases", where the solution is representable in terms of known functions. For that purpose, the parameters $\{r_0, r_1\}$ are redefined in terms of new parameters $\{\sigma, \tau\}$ according to[6]

$$r_0 = -2\,\sigma + s_1\tau, \quad r_1 = 2\,\tau \tag{19}$$

With this choice, the BHE (18) also becomes an equation explicitly depending on only two parameters $\{\sigma, \tau\}$,

$$y'' = \left(x^2 + 2\,\sigma\,x + \tau^2 + \frac{\tau}{x} + \frac{3}{4\,x^2}\right) y, \tag{20}$$

and (15) at $\{s_2 = 1, r_2 = 0, s_0 = s_1^2/4\, - 1\}$ becomes

$$y' = \frac{1}{\left(x^2 + s_1 x + \left(s_1 + 2\right)\left(s_1 - 2\right)/4\right)\, y + 2\,\tau\,x - 2\,\sigma + s_1\tau} \tag{21}$$

2.1 Liouvillian solutions for the BHE (20) when $\sigma = \pm\tau$

As explained in sec. 2.2, when $\sigma = \pm\tau$, the BHE (20) can be obtained from an equation of the form $y'' + J(x)\,y' = 0$, "missing y", through a Liouvillian transformation, and so it admits Liouvillian solutions, computable using the relatively new Kovacic algorithm [24]. Concretely, for $\sigma = \tau$, (20) becomes

$$y'' = \left(x^2 + \tau^2 + \tau\left(2\,x + \frac{1}{x}\right) + \frac{3}{4\,x^2}\right) y, \tag{22}$$

the aforementioned equation "missing y" is

$$y'' = \frac{2\,x\,(x+\tau) + 1}{x}\, y', \tag{23}$$

and the transformation mapping this equation into (22) is

$$y \rightarrow \sqrt{x}\, \mathrm{e}^{(x(x+2\,\tau)/2)}\, y \tag{24}$$

Hence a general solution for (22) is[7]

$$y = \frac{\mathrm{e}^{-x(x+2\,\tau)/2}}{\sqrt{x}}\, C_1 + \frac{\sqrt{\pi}\,\mathrm{e}^{x(x+2\,\tau)/2} - \pi\,\tau\,\mathrm{erfi}(x+\tau)\,\mathrm{e}^{-x(x+2\,\tau)/2-\tau^2}}{\sqrt{x}}\, C_2 \tag{25}$$

It is important to note that computing the transformation (24), which maps an equation in normal form, like (22), into one that is "missing y", like (23), is not a trivial operation, and is entirely equivalent to computing the solution for (22). These mappings are formally performed with the aid of infinitesimal symmetry generators, and in the case of linear ODEs, the computation of these infinitesimals indeed requires solving the ODE itself [26]. Besides the power of Kovacic's algorithm, which computes these solutions systematically just by assuming the solution field (Liouvillian), it is also remarkable that the condition $\sigma = \pm\tau$ for the existence of these Liouvillian solutions of the BHE (20) is directly evident in the canonical form of the corresponding Abel equation (29) below.

Finally, when in (20), $\sigma = -\tau$, the same treatment with Kovacic's algorithm results in the general solution

$$y = \frac{\mathrm{e}^{x(x-2\,\tau)/2}}{\sqrt{x}}\, C_1 + \frac{\sqrt{\pi}\,\mathrm{e}^{-x(x-2\,\tau)/2} - \pi\,\tau\,\mathrm{erf}(x-\tau)\,\mathrm{e}^{x(x-2\,\tau)/2+\tau^2}}{\sqrt{x}}\, C_2 \tag{26}$$

[6] The motivation for this particular choice of $\{\sigma, \tau\}$ becomes clear below, in connection with the form of equation (29).

[7] In (25) and (26), erf and erfi are respectively the error and imaginary error functions - see [25].

2.2 A solution in terms of ${}_1\mathrm{F}_1$ functions for the BHE (20) when $\sigma^2 \neq \tau^2$

A transformation relating (20) to a ${}_1\mathrm{F}_1$ differential equation, providing a solution for (20) when $\sigma^2 \neq \tau^2$, is constructed by composing three transformations: one which maps the BHE (20) into the AIR equation (21); one which maps (21) into an AIR equation admitting ${}_1\mathrm{F}_1$ solutions; and finally, one which maps that AIR equation into a ${}_1\mathrm{F}_1$ equation.

Reversing the transformations used to derive (18) from (15), the transformation mapping the Heun equation (20) into the Abel AIR (21) is

$$\left\{ x \rightarrow y, \quad y \rightarrow \frac{\mathrm{e}^{-\left(\int x\, y\, y'\, dx + \frac{s_1}{4} y^2 + \tau\, y\right)}}{\sqrt{y}} \right\} \tag{27}$$

According to [17], the transformation mapping (21) into a canonical form of AIR admitting a ${}_p\mathrm{F}_q$ solution is

$$\left\{ x \rightarrow \left(\frac{\sqrt{2}\, x}{2\,(\tau + \sigma)} + \frac{1}{2} \right)^{-1} - \frac{s_1}{2} - 1, \quad y \rightarrow \frac{\sqrt{2}\, y}{2} - \sigma \right\} \tag{28}$$

The resulting AIR canonical form is

$$y' = \frac{1}{x\, y + x^2 + (\sigma^2 - \tau^2)/2} \tag{29}$$

This form turns evident the motivation for introducing $\{\sigma, \tau\}$ according to (19). For $\sigma = \pm\tau$, the independent term in the denominator of (29) cancels, and hence, when transforming this equation into a second order linear equation, we will obtain one of the form $y'' + J(x)\, y' = 0$ with rational $J(x)$, admitting a constant for solution. Since the transformation of such an equation into a normal form like (20) is Liouvillian, the normal form of the equation will admit Liouvillian solutions.

After having determined a condition for the existence of Liouvillian solutions, for the purpose of relating the Heun equation (20) to a ${}_p\mathrm{F}_q$ equation, a simpler derivation is possible if instead of using (28) we use

$$x \rightarrow \frac{1}{x} - \frac{s_1}{2} - 1, \tag{30}$$

which does not lead to the canonical form (29), but still leads to an AIR equation admitting ${}_p\mathrm{F}_q$ solutions

$$y' = \frac{1}{(2\, x - 1)\, y + 2\,(\sigma\ + \tau)\, x^2 - 2\,\tau\, x} \tag{31}$$

Following [17], this equation is transformed into a Riccati equation, then into a linear second order one using

$$\left\{ x \rightarrow y, \quad y \rightarrow -\frac{y'}{2\,(\tau + \sigma)\, y} \right\} \tag{32}$$

leading to

$$y'' = 2\,(x - \tau)\, y' + 2\,(\tau + \sigma)\, x\, y \tag{33}$$

Finally, using $\{x \rightarrow (x + \sigma)^2,\ y \rightarrow ((x + \sigma)\, \mathrm{e}^{-x(\tau+\sigma)})^{-1}\, y\}$, equation (33) is obtained from the confluent ${}_1\mathrm{F}_1$ hypergeometric equation

$$x\, y'' + (\nu - x)\, y' - \mu\, y = 0 \tag{34}$$

at $\{\mu = (2 + \tau^2 - \sigma^2)/4,\ \nu = 3/2\}$, from where the solution to (33), in terms of the Kummer M and U functions[8] [25], is

[8] An equivalent form of this solution in terms of ${}_1\mathrm{F}_1$ functions is

$$y = \mathrm{e}^{-x(\tau+\sigma)} \left({}_1\mathrm{F}_1\left((\tau^2 - \sigma^2)/4;\ 1/2;\ (x + \sigma)^2 \right) C_1 + (x + \sigma)\, {}_1\mathrm{F}_1\left(1/2 + (\tau^2 - \sigma^2)/4;\ 3/2;\ (x + \sigma)^2 \right) C_2 \right)$$

.

$$y = \mathrm{e}^{-x(\tau+\sigma)} \left(\mathrm{M}\left(\frac{\tau^2-\sigma^2}{4}, \frac{1}{2}, (x+\sigma)^2 \right) C_1 + \mathrm{U}\left(\frac{\tau^2-\sigma^2}{4}, \frac{1}{2}, (x+\sigma)^2 \right) C_2 \right) \tag{35}$$

Summarizing, we depart from the Heun Biconfluent equation (20) and arrive at the ${}_1\mathrm{F}_1$ equation (33) with solution (35) through a process of the form

$$Heun \rightarrow Abel \rightarrow Abel_{{}_1F_1 solvable} \rightarrow {}_1\mathrm{F}_1$$

The three transformations used, (27), (30) and (32), can be combined into a single transformation, mapping the BHE (20) into the ${}_1\mathrm{F}_1$ (33) in one step:

$$y \rightarrow \frac{1}{\sqrt{x}} \exp\left(\frac{x^2}{2} - \tau x + 2(\tau+\sigma) \int \frac{x\, y}{y'}\, dx \right) \tag{36}$$

Therefore, a closed form solution for the BHE (20) when $\sigma^2 \neq \tau^2$ is given by this transformation (36), where in the "right-hand-side" the value of y is given by (35). By exploring some properties of linear differential equations discussed in sec. 5, it is possible to express the solution (36) as a linear combination of ${}_1\mathrm{F}_1$ functions with non-constant coefficients and entirely free of integrals - see (85).

An independent verification of (36), or its form free of integrals (85), as well as of the Liouvillian solutions (25) and (26), was performed in the Maple symbolic computation system - see the Appendix.

3 Closed form solutions for a subfamily of the CHE

A confluent family of Heun equations can be derived from (11) when, among the three roots ρ_i, only two are different. Hence, in (14) we have $P(y) = y$ and the starting AIR equation is

$$y' = \frac{y}{(s_2\, x^2 + s_1\, x + s_0)\, y + r_2\, x^2 + r_1\, x + r_0} \tag{37}$$

As in the previous section, changing variables using $\{x \leftrightarrow y\}$to obtain a Riccati type equation, then using (12), we obtain the linear equation

$$y'' = \frac{\left(s_2 s_1 x^2 + (s_1 r_2 + s_2 r_1)\, x - r_2 + r_2 r_1\right)}{x\,(s_2 x + r_2)}\, y' - \frac{\left(s_2 s_0 x^2 + (s_2 r_0 + s_0 r_2)\, x + r_2 r_0\right)}{x^2}\, y \tag{38}$$

This equation has two regular singularities at $\{0, -r_2/s_2\}$ and one irregular singularity at ∞, and by rewriting it in normal form, its confluent Heun type becomes evident. As shown below, the implicit restrictions in (38) as compared to the most general case (3) consist of having one of the five parameters, E, fixed and another one, B, being a function of the remaining three.

To derive the relation between the parameters of the CHE equations (3) and (38), and then a solution for (38) in the non trivial cases, we start by noting that, when $s_2 = 0$, (38) simplifies to a ${}_p\mathrm{F}_q$ equation; the interesting case is $s_2 \neq 0$, which is equivalent to taking $s_2 = 1$ in (37). The regular singularities of (38) are fixed at $\{0, 1\}$ by taking $r_2 = -1$, and the term independent of x in the normal form of (38) is fixed to be A, as in (3), by taking $s_0 = s_1^2/4 + A$. In summary, rewriting (17) in normal form, using $y \rightarrow x^{(r_1-1)/2} \sqrt{x-1} \exp(s_1\, x/2)\, y$, at $\{s_2 = 1, r_2 = -1, s_0 = s_1^2/4 + A\}$, the equation becomes

$$y'' = \left(-A + \frac{{s_1}^2/2 + s_1(r_1-1) - r_1 - 2\, r_0 + 2A + 1}{2\, x} + \frac{s_1 + r_1 - 1}{2\,(x-1)} + \frac{{r_1}^2 + 4\, r_0 - 1}{4\, x^2} + \frac{3}{4\,(x-1)^2} \right) y \tag{39}$$

which is the CHE (3) at

$$B = -A - D - C^2, \quad C = \frac{1 - s_1 - r_1}{2}, \quad D = \frac{1 - {r_1}^2}{4} - r_0, \quad E = -\frac{3}{4} \tag{40}$$

As shown in [17], the Abel AIR equation (37) can always be solved in terms of ${}_1\mathrm{F}_1$ hypergeometric functions, from where a closed form solution expressed using ${}_1\mathrm{F}_1$ can also be constructed for the CHE (39). In order to separate the Liouvillian special cases of the solutions of (39) from the generally non-Liouvillian ones, the parameters $\{r_0, r_1\}$ are redefined in terms of new parameters $\{\sigma, \tau\}$ according to $\{r_0 = (1-2\sigma)\,p^2 - {s_1}^2/4 + s_1\tau p,\ r_1 = 2\,\tau p - s_1\}$. This redefinition is derived as in the previous section, from the canonical form of the AIR (42) (see comments after (29) and also [16]). Introducing also $A = -\lambda^2$ (see [1] p.94), (39) becomes

$$y'' = \left(\lambda^2 + \frac{2\,(\sigma-1)\lambda^2 - \tau\,\lambda + 1/2}{x} + \frac{\tau\lambda - 1/2}{x-1} + \frac{(\tau^2 - 2\,\sigma + 1)\,\lambda^2 - 1/4}{x^2} + \frac{3}{4\,(x-1)^2}\right) y \tag{41}$$

that is, a 3-parameter equation, and the AIR equation (37) becomes

$$y' = \frac{y}{(x^2 + s_1 x - \lambda^2 + {s_1}^2/4)\,y - x^2 + (2\,\tau\lambda - s_1)\,x + (1-2\,\sigma)\,\lambda^2 + s_1\tau\lambda - {s_1}^2/4} \tag{42}$$

3.1 Liouvillian solutions for the CHE (41) when $\sigma = \pm\tau$

As in the previous section, we know, by construction, that at $\sigma = \pm\tau$, the CHE (41) admits Liouvillian solutions. Using Kovacic's algorithm, for $\sigma = \tau$,

$$y = \frac{x^{(1-\tau)\lambda+1/2}\,\mathrm{e}^{-\lambda\,x}}{\sqrt{x-1}} \left(C_1 + (\Gamma(2\,(\tau-1)\,\lambda+1,\ -2\,\lambda\,x) + 2\,\lambda\,\Gamma(2\,(\tau-1)\,\lambda,\ -2\,\lambda\,x))\,C_2\right) \tag{43}$$

where Γ (of two arguments - see (6.5.3) in [25]) is the incomplete gamma function. For $\sigma = -\tau$,

$$y = \frac{x^{-(1+\tau)\lambda+1/2}\,\mathrm{e}^{\lambda\,x}}{\sqrt{x-1}} \left(C_1 + (2\,\lambda\,\Gamma\,(2\,(\tau+1)\,\lambda,\ 2\,\lambda\,x) - \Gamma\,(2\,(\tau+1)\,\lambda+1,\ 2\,\lambda\,x))\,C_2\right) \tag{44}$$

3.2 A solution in terms of ${}_1\mathrm{F}_1$ functions for the CHE (41) when $\sigma^2 \neq \tau^2$

As in the previous section, a solution to (41) when $\sigma^2 \neq \tau^2$ is constructed by composing three transformations: the one which maps the CHE (41) into the AIR (42); one which maps (42) into an AIR equation admitting ${}_1\mathrm{F}_1$ solutions; finally, one which maps that AIR equation into a ${}_1\mathrm{F}_1$ equation.

Reversing the transformations used to derive (39) from (37), the transformation mapping the Heun equation (39) into the Abel AIR (42) is

$$\left\{x \rightarrow y,\quad y \rightarrow \exp\left(-\int \frac{((x+s_1/2)\,y^2 - (2\,x - \tau\,\lambda + s_1)\,y + x - \tau\,\lambda + (s_1+1)/2)\,y'}{y\,(y-1)}dx\right)\right\} \tag{45}$$

According to [17], it is possible to construct a transformation mapping (42) into a canonical form of AIR admitting a ${}_p\mathrm{F}_q$ solution. However, as in the BHE case, simpler expressions result if we transform (42) into a non-canonical AIR equation. The transformation used for this purpose is

$$x \rightarrow \frac{1}{x} - \frac{s_1}{2} - \lambda, \tag{46}$$

which maps (42) into the Abel equation

$$y' = \frac{y}{(2\,\lambda\,x - 1)\,y + 2\,\lambda^2\,(\tau+\sigma)\,x^2 - 2\,(\tau+1)\,\lambda\,x + 1} \tag{47}$$

Following [17], this AIR equation can be transformed into a ${}_p\mathrm{F}_q$ one by combining the $\{x \leftrightarrow y\}$ transformation with transformation (12) mapping a Riccati into a second order linear equation; the combination results in

$$\left\{x \to -\frac{x\,y'}{2\,(\tau+\sigma)\,\lambda^2\,y} \quad y \to x\right\} \tag{48}$$

leading to

$$y'' = \frac{2\,(x-\tau-1)\,\lambda - 1}{x}\; y' + \frac{2\,(\tau+\sigma)\,\lambda^2\,(x-1)}{x^2}\; y \tag{49}$$

Finally, using $\{x \to 2\,\lambda\,x,\; y \to x^{\left(1+\tau-\sqrt{1-2\,\sigma+\tau^2}\right)\lambda}y\}$, equation (49) is obtained from the confluent ${}_1\mathrm{F}_1$ hypergeometric equation (34) at $\{\mu = \left(\sigma - 1 + \sqrt{1-2\,\sigma+\tau^2}\right)\lambda,\; \nu = 1 + 2\,\sqrt{1-2\,\sigma+\tau^2}\,\lambda\}$, from where the solution to (49), in terms of the Whittaker $\mathbf{M}$ and $\mathbf{W}$ functions[9] [25], is

$$y = \frac{\mathrm{e}^{\lambda\,x}}{x^{(\tau+1)\lambda+1/2}} \tag{50}$$
$$\left(\mathbf{M}\left(\frac{1}{2} + (1-\sigma)\,\lambda,\; \sqrt{1-2\,\sigma+\tau^2}\,\lambda,\; 2\,\lambda\,x\right) C_1 + \mathbf{W}\left(\frac{1}{2} + (1-\sigma)\,\lambda,\; \sqrt{1-2\,\sigma+\tau^2}\,\lambda,\; 2\,\lambda\,x\right) C_2\right)$$

Summarizing, departing from the Heun confluent equation (41) we have arrived at the ${}_1\mathrm{F}_1$ equation (49) with solution (50) through a process of the form $Heun \to Abel \to Abel_{{}_1F_1 solvable} \to {}_1\mathrm{F}_1$. The three transformations (45), (46) and (48) can be combined into one transformation,

$$y \to \frac{x^{-(\tau+1)\lambda+1/2}}{\sqrt{x-1}}\;\exp\left(\lambda\,x + 2\,\lambda^2\,(\tau+\sigma)\int \frac{(x-1)\,y}{x^2 y'}\,dx\right), \tag{51}$$

which maps the CHE (41) into the ${}_1\mathrm{F}_1$ equation (49) in one step. A closed form solution for the CHE (41) when $\sigma^2 \neq \tau^2$ is then given by (51), where on the "right-hand-side" the value of y is given by (50). Like (36) in the BHE case, this solution (51) can also be expressed as a linear combination of ${}_1\mathrm{F}_1$ functions with non-constant coefficients and free of integrals - see (89). An independent check for correctness of these solutions was also performed using symbolic computation software.

4 Closed form solutions for a subfamily of the GHE

Solutions for the GHE (3) are obtained from (14) by taking $P(y) = y\,(y-1)$, that is, departing from

$$y' = \frac{y\,(y-1)}{(s_2\,x^2 + s_1\,x + s_0)\,y + r_2\,x^2 + r_1\,x + r_0} \tag{52}$$

The steps to construct these solutions are the same as those of the previous sections. Using

$$\{x \to \frac{x\,(1-x)\,y'}{(s_2 x + r_2)\,y},\; y \to x\} \tag{53}$$

the AIR equation (52) is transformed into the linear ODE

$$y'' = \frac{\left(s_2\,(s_1-1)\,x^2 + ((s_1-2)\,r_2 + s_2\,r_1)\,x + r_2\,(1+r_1)\right)}{x\,(s_2\,x+r_2)\,(x-1)}\,y' - \frac{(s_0\,x+r_0)\,(s_2\,x+r_2)}{x^2\,(x-1)^2}\,y \tag{54}$$

This is a Heun equation of the form (1), with four regular singularities at $\{0, 1, -r_2/s_2, \infty\}$. For $s_2 = 0$, (54) simplifies to a Gauss equation with ${}_2\mathrm{F}_1$ solutions. When $s_2 \neq 0$, in (52) one can take $s_2 = 1$, and, putting $r_2 = -a$, the singularities of (54) are fixed at $\{0, 1, a, \infty\}$, resulting in a non-trivial Heun family depending on four parameters. The relation between the Abel parameters $\{r_i, s_i\}$ and the Heun parameters $\{A, B, D, E, F\}$ is obtained by rewriting (54) in normal form and comparing coefficients:

[9] $\mathbf{M}\,(a, b, z) = z^{b+1/2}\,\mathrm{e}^{-z/2}\,{}_1\mathrm{F}_1(1/2 - a + b;\; 1 + 2\,b;\; z)$.

$$
\begin{aligned}
A &= r_0-(2\,r_0+s_0)\,a-\frac{1}{2}\left((r_1+1)\left(s_1+\frac{1}{a}\right)+{r_1}^2-1\right),\\
B &= \frac{r_1+s_1a+1-2\left(A\,a^2+(1-A)\,a\right)}{2\,a\,(a-1)},\\
D &= \frac{1-{r_1}^2}{4}-r_0\,a, \qquad\qquad (55)\\
E &= (1-a)\left((A+B)^2\,a-(A+B)\,(A+B-1)+\frac{D-A}{a}-\frac{D}{a^2}\right),\\
F &= -\frac{3}{4}
\end{aligned}
$$

So, with respect to the most general case (2), the restriction in the GHE (54) under consideration consists of fixing one of the six Heun parameters, F, and expressing another one, E, as a function of those remaining. To derive a solution to (54), the equation is first written in normal form and the parameters $\{s_0, r_0, r_1\}$ are redefined in terms of new parameters $\{\Delta, \sigma, \tau\}$ using

$$
s_0=\frac{{s_1}^2}{4}-\Delta^2, \qquad r_0=\left(\Delta^2-\frac{{s_1}^2}{4}\right)a-2\,\sigma\,\Delta+s_1\tau, \qquad r_1=2\,\tau-a\,s_1 \tag{56}
$$

aiming at showing that there are only four independent parameters and at separating non-Liouvillian from Liouvillian special cases of the solution, which will happen at $\sigma=\pm\tau$. The dependence of (56) on $\{\tau,\sigma\}$ is derived as in the previous sections, from the canonical form of the AIR (57) (see comments after (29) and also [16]). The new parameter Δ is related to the Heun parameter A through s_0 and the first equation in (55), and is introduced here to avoid square roots in the transformation formulas[10]. With this notation, the AIR (52) becomes

$$
y'=\frac{y\,(y-1)}{(x^2+s_1x+{s_1}^2/4-\Delta^2)\,y-a\,x^2+(2\,\tau-a\,s_1)\,x-({s_1}^2/4-\Delta^2)\,a-2\,\sigma\,\Delta+s_1\tau} \tag{57}
$$

and the Heun equation (54) in normal form, at $\{s_2=1, r_2=-a\}$, appears directly expressed in terms of the four irreducible parameters $\{a,\Delta,\sigma,\tau\}$ as

$$
\begin{aligned}
y''=\Bigg(&\frac{2\,a^2\,(a-1)\,\Delta^2-2\sigma\,a\,(2\,a-1)\,\Delta+\left(2\,\tau^2-1/2\right)a+\tau+1/2}{a\,x} \qquad\qquad (58)\\
&-\frac{2\left(a\,(a-1)^2\,\Delta^2-\sigma\,\left(2\,a-1\right)(a-1)\,\Delta+(\tau-1/2)\,((\tau+1/2)\,a-\tau)\right)}{(a-1)\,(x-1)}+\frac{\tau-a+1/2}{a\,(a-1)\,(x-a)}\\
&+\frac{a^2\Delta^2-2\,a\,\sigma\,\Delta+\tau^2-1/4}{x^2}+\frac{(a-1)^2\,\Delta^2-2\,\sigma\,\left(a-1\right)\Delta+\tau^2-1/4}{(x-1)^2}+\frac{3}{4\,\left(x-a\right)^2}\Bigg)\,y
\end{aligned}
$$

4.1 Liouvillian solutions for the GHE (58) when $\sigma=\pm\tau$

By construction, as in the previous sections, for $\sigma=\pm\tau$, the GHE (58) admits Liouvillian solutions computable using Kovacic's method; for $\sigma=\tau$,

[10] The use of $\lambda^2=-A$ in sec. 3 brings the same advantage.

$$
y = \frac{x^{\tau - a\Delta + 1/2}\,(x-1)^{(a-1)\Delta - \tau + 1/2}}{\sqrt{a-x}} \tag{59}
$$

$$
\left(C_1 + \left(\mathbf{B}_x\left(1 + 2\,(a\,\Delta - \tau)\,,\,2\,((1-a)\,\Delta + \tau)\right) - a\,\mathbf{B}_x\left(2\,(a\,\Delta - \tau)\,,\,2\,((1-a)\,\Delta + \tau)\right)\right) C_2\right)
$$

where $\mathbf{B}_x$ (one index and two arguments - see (6.6.1) in [25]) is the incomplete beta function. The Liouvillian form of the solution of (58) at $\sigma = -\tau$ is

$$
y = \frac{x^{\tau + a\Delta + 1/2}\,(x-1)^{(1-a)\Delta - \tau + 1/2}}{\sqrt{a-x}} \tag{60}
$$

$$
\left(C_1 + C_2\left(\mathbf{B}_x\left(1 - 2\,(a\,\Delta + \tau)\,,\,2\,((a-1)\,\Delta + \tau)\right) - a\,\mathbf{B}_x\left(-2\,(a\,\Delta + \tau)\,,\,2\,((a-1)\,\Delta + \tau)\right)\right)\right)
$$

The existence of solutions in terms of the incomplete Beta functions is also pointed out in [14], where a solution for the GHE as a power series is constructed, and an approximate solution involving a combination of two incomplete Beta functions is derived - see also [10].

4.2 A solution in terms of ${}_2\mathrm{F}_1$ functions for the GHE (58) when $\sigma^2 \neq \tau^2$

As in the case of the BHE and CHE equations, a solution for the GHE (58) when $\sigma^2 \neq \tau^2$ is constructed by composing a transformation mapping (58) into the Abel AIR (57) with a transformation mapping (57) into a second order linear equation admitting hypergeometric solutions, in this case of ${}_2\mathrm{F}_1$ type.

The derivation of results below follows the same path shown in the previous sections. Summarizing, the transformation mapping the GHE (58) being solved into the AIR (57) is

$$
\left\{x \rightarrow y,\; y \rightarrow \mathrm{e}^{\int \frac{\left((x+(s_1-1)/2)y^2+((1-2\,x-s_1)a+\tau)y+((s_1/2+x)a-\tau-1/2)a\right)y'}{y(y-1)(a-y)}\,dx}\right\} \tag{61}
$$

The transformation mapping the AIR (57) into a ${}_p\mathrm{F}_q$ second order linear equation is

$$
\left\{x \rightarrow -2\,\frac{\Delta\;(\tau+\sigma)\,y}{x\,(x-1)\,y'} - \frac{s_1}{2} - \Delta, \quad y \rightarrow x\right\} \tag{62}
$$

and the resulting equation, of ${}_2\mathrm{F}_1$ type, is

$$
y'' = \frac{(2\;(\Delta-1)\,x + 1 - 2\,(a\,\Delta+\tau))}{x\,(x-1)}\,y' + \frac{2\;(x-a)\,\Delta\;(\tau+\sigma)}{x^2\,(x-1)^2}\,y \tag{63}
$$

Combining the transformations (61) and (62), a transformation mapping (58) into (63) in one step is

$$
y \rightarrow \frac{x^{a\,\Delta+\tau+1/2}\;(x-1)^{(1-a)\Delta-\tau+1/2}}{\sqrt{a-x}}\;\exp\left(2\,\Delta\;(\tau+\sigma)\int\frac{(x-a)\,y}{(x-1)^2\,x^2\,y'}\,dx\right) \tag{64}
$$

Hence, the solution to the Heun equation (58) to which this section is dedicated is given by the expression above, where, in the "right-hand-side", the value of y is given by the solution to (63), that is,

$$
y = x^{a\Delta+\tau-\mathrm{T}}\;(x-1)^{(1-a)\,\Delta+\Sigma-\tau} \tag{65}
$$

$$
\left({}_2\mathrm{F}_1(\Sigma+\Delta-\mathrm{T},\Sigma-\Delta+1-\mathrm{T};\,1-2\,\mathrm{T};\,x)\,C_1 + x^{2\,\mathrm{T}}\,{}_2\mathrm{F}_1(\Sigma+\Delta+\mathrm{T},\Sigma-\Delta+1+\mathrm{T};\,1+2\,\mathrm{T};\,x)\,C_2\right)
$$

where, to make the structure of this solution visible, instead of $\{\sigma,\tau\}$ we are using

$$\Sigma = \sqrt{(a-1)^2\,\Delta^2 - 2\,(a-1)\,\sigma\,\Delta + \tau^2}, \quad \mathrm{T} = \sqrt{a^2\Delta^2 - 2\,a\,\sigma\,\Delta + \tau^2} \tag{66}$$

This solution (65) in turn is computed noting that (63) is obtained by changing variables

$$y \rightarrow x^{\mathrm{T} - a\,\Delta - \tau}\,(x-1)^{\tau - \Sigma + (a-1)\Delta}\,y \tag{67}$$

in Gauss' ${}_2\mathrm{F}_1$ equation

$$\left(x^2 - x\right) y'' + ((\mu + \nu + 1)\,x - \rho)\,y' + \mu\,\nu\,y = 0 \tag{68}$$

taken at $\{\mu = \Sigma + \Delta - \mathrm{T},\ \nu = \Sigma - \Delta + 1 - \mathrm{T},\ \rho = 1 - 2\,\mathrm{T}\}$. Hence, the same transformation (67) maps the solution of Gauss' equation into (65). For a solution equivalent to (64), free of integrals, expressed as a linear combination of ${}_2\mathrm{F}_1$ functions with non-constant coefficients, see (92). These and the Liouvillian solutions presented for the GHE (58) were also verified for correctness using symbolic computation software.

5 Alternative derivation of solutions free of integrals

In the previous sections, non-Liouvillian solutions, as well as their Liouvillian special cases and the relationship between the Heun parameters for their existence, were derived for the BHE, CHE and GHE equations (20), (41) and (58). The non-Liouvillian solutions (36), (51) and (64), however, have the drawback of containing non-trivial uncomputed integrals. In this section, from the knowledge of the form of these solutions and exploring non-local transformations, equivalent solutions free of integrals are derived.

We start by recalling that second order linear equations

$$y'' = c_1\,y' + c_0\,y \tag{69}$$

where $c_i \equiv c_i(x)$, can always be mapped into Riccati equations (10) back and forth. The transformation mapping (69) into a Riccati equation is of the form

$$y \rightarrow \mathrm{e}^{-\int G(x)\,y\,dx} \tag{70}$$

where $G(x)$ is an arbitrary function, and the transformation mapping a Riccati equation (10) into a linear equation is given by (12). It is also known that any two Riccati equations can be mapped between themselves through Möbius transformations of the dependent variable y with variable coefficients $f_i \equiv f_i(x)$,

$$y \rightarrow \frac{f_1\,y + f_2}{f_3\,y + f_4}, \tag{71}$$

where $f_1 f_4 - f_3 f_2 \neq 0$. If we now transform (69) into a Riccati equation using (70), then apply the Möbius transformation (71), and to the resulting equation we apply transformation (12), we obtain another second order linear equation. The composition of these three transformations is the non-local transformation

$$y \rightarrow \exp\left(-\int \frac{f_1 \Omega\,y' + f_2 \Upsilon\,y}{f_3 \Omega\,y' + f_4 \Upsilon\,y}\,dx\right), \tag{72}$$

where

$$\Omega = f_1 f_4 - f_3 f_2 \neq 0, \qquad \Upsilon = c_0\,{f_3}^2 + ({f_1}' - c_1 f_1)\,f_3 - ({f_3}' + f_1)\,f_1 \neq 0, \tag{73}$$

and, in fact, this transformation suffices to generate the whole class of linear equations from any given one. For example, applying (72) at $f_1 = f_4 = 1$ to $y'' = 0$, we obtain an equation as general as (69).

The particular case of (72) at $f_1 = f_4 = 0$ and $f_2 = f_3 = 1$,

$$y \rightarrow \exp\left(\int \frac{c_0\,y}{y'} dx\right) \tag{74}$$

is relevant to the results of the previous sections: the transformations (36), (51) and (64), respectively mapping the BHE, CHE and GHE into ${}_p\mathrm{F}_q$ equations, are in fact compositions of transformations of the form (74) with transformations $y \to P(x)\, y$. The linear equation obtained by applying (74) to (69) is

$$y'' = \left(\frac{{c_0}'}{c_0} - c_1\right) y' + c_0\, y \tag{75}$$

By applying to this equation the same transformation (74), we reobtain[11] (69). So, y in the "left-hand-side" of (74) represents the solution to (75) or (69), respectively written in terms of the solution to (69) or (75), represented by y in the "right- hand-side" of (74). As we shall see in the following subsections, this is indeed the mechanism by which solutions to Heun equations (here represented by (75)) were expressed as exponentials of integrals of solutions to ${}_p\mathrm{F}_q$ equations (here represented by (69)) in the previous sections.

With this understanding of matters, however, it is possible to show that, despite the integral sign entering (74), the solution to (75) can be derived without performing any integration. For that purpose, we note that, given a generic linear ODE in y, it is always possible to construct the generic linear ODE of the same order satisfied by $p = y'$. Concretely, if y satisfies (69), the ODE for p is[12]

$$p'' = \left(\frac{{c_0}'}{c_0} + c_1\right) p' + \left({c_1}' + c_0 - \frac{{c_0}' c_1}{c_0}\right) p \tag{76}$$

and by substituting $y = \int p \; dx$ into (69), we obtain

$$y = \int p \; dx = \frac{p' - c_1\, p}{c_0} \tag{77}$$

Therefore, when (76) can be solved, the solution y for (69) can be obtained directly from p by differentiation. For instance, writing p in terms of some $f \equiv f(x)$ and $g \equiv g(x)$, as

$$p = f\, C_1 + g\, C_2, \tag{78}$$

a solution for (69) is computed from p, without using integration, as

$$y = \frac{f\,' - c_1\, f}{c_0}\, C_1 + \frac{g\,' - c_1\, g}{c_0}\, C_2 \tag{79}$$

Composing now the non-local transformation (74) with the introduction of $p = y'$, that is, plugging the coefficients of (75) into (76), we obtain

$$p'' = \left(\frac{2\, {c_0}'}{c_0} - c_1\right) p' + \left(c_0 - {c_1}' + \frac{c_1 {c_0}' + {c_0}''}{c_0} - 2\left(\frac{{c_0}'}{c_0}\right)^2\right) p, \tag{80}$$

and the key observation is that the normal form[13] of (80) is the same as that of (69). Consequently, if (69) is of ${}_p\mathrm{F}_q$ type[14], then (80) is too, even when (75) may not be (in what follows it will be of Heun type), and the solution to (75) can be expressed not just using the integral form (74), but also using (79) as a linear combination, with variable coefficients, of the ${}_p\mathrm{F}_q$ solutions $\{f, g\}$ of (80) and their derivatives.

Based on these observations, the derivation of the solutions (36), (51) and (64) for the BHE, CHE and GHE equations (20), (41) and (58), performed in the previous sections, can be reformulated entirely, shortcutting the Abel equation step, and resulting in solutions free of integrals as follows.

[11] The composition of (74) with itself is equal to the identity transformation.

[12] In the easier case $c_0 = 0$, the equation for p is: $p'' = c_1 p' + {c_1}' p$.

[13] For rewriting equations in normal form, see the Appendix.

[14] An equation is of ${}_p\mathrm{F}_q$ type if it admits solutions of the form (8); these solutions can be computed systematically [18].

5.1 Solution free of integrals for the BHE (20)

Applying transformation (74) to the ${}_1\mathrm{F}_1$ equation (33), derived from the AIR (31) in sec. 2, we obtain

$$y'' = \left(2\,(\tau - x) + \frac{1}{x}\right) y' + 2\,(\tau + \sigma)\,x\,y \tag{81}$$

This equation is already of Heun type, and by rewriting it in normal form[15], that is, changing further

$$y \to \exp\left(\int \frac{1}{2\,x} - x + \tau\,dx\right) y \tag{82}$$

we directly obtain the two parameter BHE (20), derived from the AIR (15). Hence, combining these two transformations (74) and (82), we also directly obtain the solution (36) computed in sec. 2 for the BHE (20).

We note here that the particular form (33) of the ${}_1\mathrm{F}_1$ equation derived in sec. 2, which has only one irregular singularity at infinity, has the important feature that under the transformation (74), (33) gains one regular singularity at the origin. Hence, the resulting equation (81) is not a ${}_1\mathrm{F}_1$ equation anymore but a 2-parameter biconfluent Heun equation. Both the augmentation in the number of singularities under transformation (74) and the change in type from ${}_1\mathrm{F}_1$ to BHE do not happen with all ${}_1\mathrm{F}_1$ equations.

Now, since this BHE equation (81) was obtained from a ${}_1\mathrm{F}_1$ equation using the transformation (74), as explained, the derivative $p \equiv y'$ of the solution to (81) also satisfies a ${}_1\mathrm{F}_1$ equation. According to (76), the equation for $p = y'$ associated to (81) is

$$p'' = \frac{2\left(1 + \tau\,x - x^2\right)}{x}\,p' - \frac{2\left(1 + \tau\,x - (\tau + \sigma)\,x^3\right)}{x^2}\,p \tag{83}$$

The solution to this ${}_1\mathrm{F}_1$ equation can be expressed in terms of the Kummer functions M and U [25] as

$$p = x\,\mathrm{e}^{x(\tau - \sigma - x)}\left(\mathrm{M}\left(\frac{\tau^2 - \sigma^2}{4},\,\frac{1}{2},\,(x+\sigma)^2\right) C_1 + \mathrm{U}\left(\frac{\tau^2 - \sigma^2}{4},\,\frac{1}{2},\,(x+\sigma)^2\right) C_2\right) \tag{84}$$

Substituting this solution into (79), we obtain the solution to the BHE (81), and further applying the transformation (82), we directly obtain the general solution, free of integrals, for the BHE (20) of sec. 2, as

$$\begin{aligned} y = \frac{\mathrm{e}^{-\sigma\,x - x^2/2}}{\sqrt{x}\,(x+\sigma)} \Bigg(& \left(\Lambda(x)\,\mathrm{U}\left(\frac{\tau^2-\sigma^2}{4},\,\frac{1}{2},\,(x+\sigma)^2\right) - 4\,\mathrm{U}\left(\frac{\tau^2-\sigma^2}{4} - 1,\,\frac{1}{2},\,(x+\sigma)^2\right)\right) C_1 \\ + & \left(\left(\tau^2 - \sigma^2 - 2\right)\mathrm{M}\left(\frac{\tau^2-\sigma^2}{4} - 1,\,\frac{1}{2},\,(x+\sigma)^2\right) - \Lambda(x)\,\mathrm{M}\left(\frac{\tau^2-\sigma^2}{4},\frac{1}{2},(x+\sigma)^2\right)\right) C_2 \Bigg) \end{aligned} \tag{85}$$

where $\Lambda(x) \equiv \sigma^2 + \tau^2 + 2\,(2\,x^2 + (3\,\sigma - \tau)\,x - \sigma\,\tau - 1)$ and $\sigma^2 \neq \tau^2$. Symbolic computation input for verifying this solution is found in the Appendix.

This approach to the solution of the BHE (20) is straightforward, clearly simpler than the calculations presented in sec. 2. As shown in the following two subsections, this simpler approach also leads to solutions free of integrals for the CHE (41) and the GHE (58) discussed in sec. 3 and sec. 4. We note, however, that without the knowledge of the equations of Heun and ${}_p\mathrm{F}_q$ type being linked, or of the form of the transformation relating them[16], both of which were derived in the previous sections from the connection Heun $\leftrightarrow$ Abel, the existence of the straightforward mechanism used in this section is not evident.

[15] For the formula to write linear equations in normal form, see the Appendix.
[16] In the BHE case, these are (20), (33) and (36).

5.2 Solution free of integrals for the CHE (41)

Like in the BHE case, a solution to the CHE (41), equivalent to (51) and free of integrals, can be obtained by first applying (74) to the ${}_1\mathrm{F}_1$ equation (49), derived from the AIR (47) in sec. 3, leading to a CHE,

$$y'' = \frac{1 + 2\,\lambda\,(x-1)\,(1-\tau-x)}{x\,(x-1)}\,y' + \frac{2\,\lambda^2\,(x-1)\,(\tau+\sigma)}{x^2}\,y \tag{86}$$

Rewriting this equation in normal form directly results in the CHE (41) discussed in sec. 3. So, the problem now is the computation of solutions to the CHE (86). As in the BHE case, we know, by construction, that when y satisfies (86), $p \equiv y'$ satisfies a ${}_1\mathrm{F}_1$ equation. According to (76), the equation for p is

$$\begin{aligned} p'' = {} & \frac{3 - 2\,\lambda\,(1+\tau) + (2\,\lambda\,(\tau+2) - 1)\,x - 2\,\lambda\,x^2}{x\,(x-1)}\,p' \\ + {} & \frac{2\,\lambda^2\,(x-1)^3\,(\tau+\sigma) - 2\,(x-1)\left(x^2 - 2\,x + \tau + 1\right)\lambda - 1 - x}{x^2\,(x-1)^2}\,p \end{aligned} \tag{87}$$

The general solution to this equation can be written in terms of Whittaker functions $\mathbf{M}$ and $\mathbf{W}$ [25] as

$$p = \frac{(x-1)\,x^{(\tau+1)\lambda - 3/2}}{\mathrm{e}^{\lambda\,x}}\left(\mathbf{M}\left(\mu,\,\nu,\,2\,\lambda\,x\right)\,C_1 + \mathbf{W}\left(\mu,\,\nu,\,2\,\lambda\,x\right)\,C_2\right) \tag{88}$$

where $\mu = \lambda\,(1-\sigma) + 1/2$ and $\nu = \lambda\,\sqrt{\tau^2 - 2\,\sigma + 1}$. Substituting this solution into (79) leads to a solution free of integrals for the CHE (86), from where the solution to its normal form (41) of sec. 3, is

$$\begin{aligned} y = \frac{1}{\sqrt{x-1}} \Bigg(& \left(\mathbf{W}\left(\mu,\,\nu,\,2\,\lambda\,x\right) + \lambda\,(\tau-\sigma)\,\mathbf{W}\left(\mu-1,\,\nu,\,2\,\lambda\,x\right)\right) C_2 \\ & + \left(\lambda\,(\tau+\sigma)\,\mathbf{M}\left(\mu,\,\nu,\,2\,\lambda\,x\right) + ((1-\sigma)\,\lambda - \nu)\,\mathbf{M}\left(\mu-1,\,\nu,\,2\,\lambda\,x\right)\right) C_1 \Bigg) \end{aligned} \tag{89}$$

Symbolic computation input for verifying this solution is found in the Appendix.

5.3 Solution free of integrals for the GHE (58)

The derivation done in sec. 4 of a solution for the GHE (58) can be reformulated as in the BHE and CHE cases. Applying (74) to the ${}_2\mathrm{F}_1$ hypergeometric equation (63), we obtain

$$y'' = \frac{(2\,\Delta+1)\,x^2 - 2\,((2\,\Delta+1)\,a + \tau)\,x + 2\,a^2\Delta + (2\,\tau+1)\,a}{x\,(x-1)\,(a-x)}\,y' - \frac{2\,(a-x)\,\Delta\,(\tau+\sigma)\,y}{x^2\,(x-1)^2} \tag{90}$$

which is a GHE with four regular singularities at $\{0, 1, a, \infty\}$. Rewriting (90) in normal form using

$$y \rightarrow \exp\left(\int \frac{2\,(x-a)\,\tau - 2\,(a-x)^2\,\Delta + a\,(2\,x-1) - x^2}{2\,x\,(x-1)\,(x-a)}\,dx\right) y \tag{91}$$

we obtain the 4-parameter GHE (58) of sec. 4. The solution (64) presented in sec. 4 for this equation is identical to the composition of these two transformations (74) and (91). As in the previous subsections, using (76) we compute the equation for $p = y'$ associated to (90), which, as discussed, is by construction a ${}_2\mathrm{F}_1$ equation when written in normal form. Solving for p, from (79), we obtain the solution to (90), and applying (91) to it, we obtain a solution free of uncomputed integrals for the GHE (58),

$$
y = \frac{(x-1)^{\Sigma+1/2}}{\sqrt{x-a}} \Bigg(\tag{92}
$$

$$
C_1 \left(\frac{(\mathrm{T}-\Sigma-\Delta)(\mathrm{T}-\Sigma+\Delta-1)(x^{5/2-\mathrm{T}}-x^{3/2-\mathrm{T}})}{2} \, {}_2\mathrm{F}_1(\Sigma+\Delta-\mathrm{T}+1, \Sigma-\Delta-\mathrm{T}+2;\, 2\,(1-\mathrm{T});\, x) \right.
$$

$$
\left. + \left(\mathrm{T}-\frac{1}{2}\right)\left((a\Delta-\mathrm{T}+\tau)\, x^{1/2-\mathrm{T}} + (\mathrm{T}-\Sigma-\Delta)\, x^{3/2-\mathrm{T}}\right) {}_2\mathrm{F}_1(\Sigma+\Delta-\mathrm{T}, \Sigma-\Delta-\mathrm{T}+1;\, 1-2\,\mathrm{T};\, x) \right)
$$

$$
+ C_2 \left(\frac{(\mathrm{T}+\Sigma+\Delta)(\mathrm{T}+\Sigma-\Delta+1)(x^{3/2+\mathrm{T}}-x^{5/2+\mathrm{T}})}{2} \, {}_2\mathrm{F}_1(\Sigma+\Delta+\mathrm{T}+1, \Sigma-\Delta+\mathrm{T}+2;\, 2\,(1+\mathrm{T});\, x) \right.
$$

$$
\left. + \left(\mathrm{T}+\frac{1}{2}\right)\left((a\Delta+\mathrm{T}+\tau)\, x^{1/2+\mathrm{T}} - (\mathrm{T}+\Sigma+\Delta)\, x^{3/2+\mathrm{T}}\right) {}_2\mathrm{F}_1(\Sigma+\Delta+\mathrm{T}, \Sigma-\Delta+1+\mathrm{T};\, 1+2\,\mathrm{T};\, x) \right) \Bigg)
$$

where Σ and T are defined in (66). This solution is also valid when the GHE (58) admits Liouvillian solutions, that is, when $\sigma^2 = \tau^2$, although in this case the solutions (59) and (60) are expressed in simpler manner. Symbolic computation input for verifying this solution (92) is found in the Appendix.

6 Comparison with solutions existing in the literature

A search in the literature didn't show previous references to a link between Heun and Abel equations as the one presented in sec. 1, nor a derivation of solutions to the former equations and confluent cases from the knowledge of solutions to the latter. It is nonetheless interesting to compare the solutions for Heun equations derived through this link *Heun* $\leftrightarrow$ *Abel* and in sec. 5 with the ones previously presented in the literature. For practical reasons, the discussion is restricted to three more recent papers, by Ronveaux [9], by Ishkhanyan and Suominen [10], and by Shanin and Craster [15], which present sufficiently explicit solutions for the GHE (1), similar to the non-Liouvillian solutions and the special Liouvillian cases derived in sec. 4.1 and sec. 5.3.

6.1 Liouvillian solutions

In [9], the factorization of Heun's General equation (1) into a form

$$
(L(x)\mathrm{D} + M(x))\left(\bar{L}(x)\mathrm{D} + \bar{M}(x)\right) y = 0 \tag{93}
$$

where $D \equiv d/dx$ and $\{L, M, \bar{L}, \bar{M}\}$ are polynomials, is discussed, and six sets of conditions on the Heun parameters, such that this type of factorization is possible, are derived, all leading to solutions of the form

$$
y = x^{\rho_1}\,(x-1)^{\rho_2}\,(x-a)^{\rho_2} \tag{94}
$$

for some ρ_i. Since these solutions are Liouvillian, they can be computed systematically, e.g., in a symbolic computation environment like Maple or Mathematica, where Kovacic's algorithm is implemented.

The Liouvillian solutions (59) and (60), derived here from the condition $\sigma^2 = \tau^2$ related to the canonical form of Abel equations, are also of the form (94). Nonetheless, the conditions for the existence of a factorization of the form (93) obtained in [9] are less general than the conditions for the existence of Liouvillian solutions derived here. For example, if in (58) we change variables $y \to \exp(y)$, the resulting Heun equation will continue having rational coefficients and the condition $\sigma^2 = \tau^2$ will continue assuring that the solution admits Liouvillian form, computable using Kovacic's algorithm, even when it won't be of the form (94) anymore. On the other hand, if we perform the same change of variables in the equations obtained in [9], the resulting equations will be out of reach of the factorization there presented, because $M(x)$ in (93) will have an exponential factor, while that method applies only to polynomial forms of $M(x)$.

We note that, by changing variables appropriately, the Liouvillian solutions derived in sec. 3.1 and 4.1 for the BHE (20) and CHE (41), can also be transformed into the form (94), which indicates that a factorization like the one discussed in [9] exists also for the BHE and CHE.

6.2 Non-Liouvillian solutions

Non-Liouvillian solutions cannot be computed with Kovacic's algorithm, nor is there such a general algorithm for computing them. This type of solutions was presented in the previous sections and is discussed in other papers in the literature.

In [10], an approach restricted to the GHE[17] (1) is discussed. Concretely, after some manipulations, the equation satisfied by H', where H is a solution to the Heun equation (1), is presented. This equation for H' can be computed using (76), and is a Fuchsian equation with five regular singularites, located at $\{0, 1, \infty, a, q/(\alpha\,\beta)\}$. So, when $q/(\alpha\,\beta)$ is equal to 0, 1 or a, or approaches ∞, the equation has four singularities and hence both H and H' satisfy a Heun equation. This happens when either $q = 0$, $q = \alpha\,\beta$, $q = a\,\alpha\,\beta$, or $\alpha\,\beta = 0$. These four cases are presented in [10], and at first sight, the approach could be compared with the one presented here, in sec. 5, where Heun and related confluent equations with the property that H' is of the form (8), involving ${}_p\mathrm{F}_q$ functions, were derived.

The main difference between the presentation in sec. 5 and that in [10] is that, in sec. 5, the non-local transformation (74) directly leads to non-trivial Heun equations in H, such that H' can be computed *systematically*, because it is of the form (8), and from there we can systematically compute H, using (79).

On the other hand, in [10], the condition that H' satisfies a Heun equation of one parameter less is not sufficient to compute its value. So, to obtain solutions using this approach, the authors introduce additional restrictions on the values of the Heun parameters so that H' admits solutions expressible using ${}_2\mathrm{F}_1$ functions. Neither the origin of these ad-hoc restrictions nor a systematic manner of computing them is shown.

For the first case, $q = 0$, the additional restrictions suggested in [10] are $\epsilon = -1$ and $q' = a\,\alpha'\beta'$, where $\{q', \alpha', \beta'\}$ are some functions of the Heun parameters $\{\alpha, \beta, \gamma, \delta, \epsilon, q\}$, together leading to

$$H' = x\ {}_2\mathrm{F}_1(\alpha', \beta';\, \gamma + 2;\, x) \tag{95}$$

This is a case depending on only three parameters $\{\alpha', \beta', \gamma\}$, which happens to be a particular case of the 4-parameter GHE solved here, in sec. 5.3. That can be seen by using (76) to construct the equation for $p \equiv H'$ associated to Heun equation (90) of sec. 5.3, and by noting that its solutions are of the form (8)

$$p = H' = x^{\rho_1}(x-1)^{\rho_2}(x-a)^{\rho_3}\ {}_2\mathrm{F}_1(\tilde\alpha, \tilde\beta;\, \tilde\gamma;\, x) \tag{96}$$

for some ρ_i; that is, they depend on four parameters $\{a, \tilde\alpha, \tilde\beta, \tilde\gamma\}$, not three.

The other case explicitly discussed in [10] is $q = \alpha\,\beta$, so a 5-parameter GHE for which a solution is shown in terms of an infinite series of Appel functions[18] [28] which, in one case, is shown to be expressible as an infinite sum of ${}_2\mathrm{F}_1$ functions, presented in [10] with number (36). It is not evident how to compare this formal infinite series solution with the finite-number-of-terms solutions presented here, in sec. 5.3, nor is it evident how to impose an additional restriction to the 5-parameter GHE treated in [10] such that the formal infinite series expansion terminates. Solutions to the GHE in terms of infinite series of ${}_2\mathrm{F}_1$ functions are also known in the literature [1], although the authors of [10] make the point that the ${}_2\mathrm{F}_1$ functions entering the formal series they present have a different behavior than those shown in [1].

An approach based on removing "false" singularities

A referee has also pointed to [15], a very interesting paper by Shanin and Craster, published in 2002 but actually submitted during 2000, where a more thorough approach to solving GHE and confluent CHE equations (actually, the approach is for all linear equations with "false" singularities), is presented.

The first idea in [15] consists of determining relations (constraints) between the Heun parameters such that one of the regular singularities is "false". That will happen when the corresponding characteristic exponents differ by an integer but, also, no logarithmic term appears in the local expansion of the solution. Although such an approach is entirely different from the one developed here, where the solvable Heun equations are derived from the single Abel AIR (11), it is remarkable that the BHE (20), CHE (41) and GHE (58) resulting from this link $Heun \leftrightarrow Abel$ do have one such false singularity.

[17] The BHE, CHE or other confluent cases are not discussed in [10].

[18] The Appel function is a formal extension of the ${}_2\mathrm{F}_1$ function to two variables, expressed as a double infinite series.

In [15], there is no explicit discussion of the BHE (4), but the approach seems applicable to that case too, and, as is the case here, the approach does not seem to be applicable to the DHE (5) or the THE (6). The derivation of GHE and CHE solvable equations in [15] is systematic but not as straightforward as the one-step derivation shown here in sec. 5.2 and sec. 5.3, exploring non-local transformations. Concretely, in [15], determining the value of the accessory parameter q in (1), such that the equation has a false singularity, requires using ${}_p\mathrm{F}_q$ identities and solving recurrence relations for the coefficients of series expansions, to assure there is no logarithmic term in the solution.

The second idea presented in [15] is that, when the Heun equation has a false singularity, its solution can be expressed as a linear combination (with finite number of terms and constant coefficients) of ${}_p\mathrm{F}_q$ functions, determined by exploring isomonodromy mappings. Although finding the appropriate isomonodromy is a powerful idea, and the approach is systematic, quoting the authors of [15] (p.628): *"The procedure of finding an appropriate isomonodromy mapping described is quite complicated. In the relatively simple examples that we construct we can pose an ansatz for the form of the mapping to within several unknown parameters; these are found by direct substitution."*. The actual procedure to determine these unknown parameters is systematic, but not so straightforward; quoting the authors (p.630): *"The simplest way to do this is to substitute the linear combination, say $U + C\,V$ directly into the Heun equation and then use known recursion formulae for hypergeometric functions and their derivatives. Tedious calculations show that ... "*.

In contrast, the approach presented here, in sec. 5, directly leads, by construction, to the three multiparameter BHE (81), CHE (86) and GHE (90) equations having false singularities as well as to the exact form of the linear combinations (with non-constant coefficients) of ${}_p\mathrm{F}_q$ functions that solve these equations.

Furthermore, here, in sec. 5, the use of non-local transformations links ${}_p\mathrm{F}_q$ equations also to other linear equations, with more singularities than those of the Heun families, where again the linear combinations (79) solving all these equations involve non-constant coefficients. Although these cases can in principle be treated by finding isomonodromies, that approach may result non-practical as soon as the number of singularities or the number of parameters involved increases. For example, in perhaps the simplest case, departing from

$$\frac{{}_0\mathrm{F}_1(\ ;\, a;\, x)}{x-\kappa} \tag{97}$$

the ${}_p\mathrm{F}_q$ equation satisfied by this expression is

$$y'' = \left(-\frac{a}{x} + \frac{2}{\kappa - x}\right) y' + \frac{(x-\kappa-a)\,y}{x\,(x-\kappa)} \tag{98}$$

Applying now the non-local transformation (74) we obtain

$$y'' = \left(\frac{a-1}{x} + \frac{1}{x-\kappa} + \frac{1}{x-a-\kappa}\right) y' + \frac{(x-\kappa-a)\,y}{x\,(x-\kappa)} \tag{99}$$

This equation has three regular singularities at $\{0, \kappa, a+\kappa\}$ and one irregular singularity at ∞; they are all irreducible, and therefore (99) does not fit into any of the five Heun classes represented by (2-6) (it belongs to an "upper" class). According to sec. 5, by construction, the equation satisfied by $p = y'$ admits systematically computable [18] solutions of the form (8) which, when plugged into (79), lead to the following solution to (99):

$$\begin{aligned} y = {} & C_1\, x^a \left(a\; {}_0\mathrm{F}_1(\ ;\, a;\, x) - (x-\kappa)\, {}_0\mathrm{F}_1(\ ;\, a+1;\, x)\right) \\ & + C_2 \left((a-2)\left((1-a)\,\kappa + a\,x\right) {}_0\mathrm{F}_1(\ ;\, 2-a;\, x) + x\,(x-\kappa)\, {}_0\mathrm{F}_1(\ ;\, 3-a;\, x)\right) \end{aligned} \tag{100}$$

This approach, as described in sec. 5, works just as straightforwardly when we start with ${}_p\mathrm{F}_q$ functions more general than (97), while through that process the equation resulting from applying (74) can be made to depend on more parameters and have more singularities. Even so, by construction, the exact linear combination (with non-constant coefficients) of ${}_p\mathrm{F}_q$ functions solving the resulting equation is always given by (79). Contrasting with that, depending on the starting ${}_p\mathrm{F}_q$ expression to be used in place of (97), the

construction of the same solvable cases and computation of their solutions using the approach presented in [15] can be really complicated.

On the other hand, an important generalization presented in [15] is that it provides a recipe for computing the isomonodromies and constructing the related Heun equations having as solutions linear combinations of ${}_p\mathrm{F}_q$ functions involving more than two terms.

7 Discussion

In sec. 2, 3 and 4, solutions in terms of ${}_p\mathrm{F}_q$ functions were derived for families of the Heun equations BHE, CHE and GHE. The approach links linear equations with four regular singularites (and related confluent cases) to linear equations with three regular singularities (and related confluent cases), by linking both types of linear equations to the canonical forms of the Abel AIR class of non-linear first order equations. The link $AIR \leftrightarrow {}_p\mathrm{F}_q$ is developed in [17], and the link $AIR \leftrightarrow Heun$ is presented in sec. 1. This link also provided a natural way to determine the special Liouvillian cases of the Heun solutions to the BHE, CHE and GHE here treated, and permits studying Abel equation problems by reformulating them in terms of linear equations.

In sec. 5, that approach is shown to be equivalent to performing the non-local transformation (74) on the ${}_p\mathrm{F}_q$ equations (33), (49) and (63), leading to Heun equations with two important properties: 1) further auxiliary equations which can be derived from them for $p \equiv y'$ are of ${}_p\mathrm{F}_q$ type; 2) when written in normal form, these Heun equations obtained using (74) are identical to the BHE (20), CHE (41) and GHE (58) solved in the sections previous to sec. 5. This approach leads in a simpler manner to the same solutions (36), (51) and (64), and also to the equivalent forms of these solutions free of integrals, (85), (89) and (92). All the solutions presented were verified for correctness using symbolic computation software.

Besides the presentation in sec. 2, 3 and 4, the existence of a connection between the "Heun and related confluent equations" on the one hand and the "AIR (11) and the different possible multiplicities of its roots ρ_i" on the other hand, can be seen more straightforwardly by transforming not (14) but (11) into a linear equation[19], resulting in

$$y'' = \left(\frac{1}{x-a} + \frac{R_3}{x-\rho_3} + \frac{R_2}{x-\rho_2} + \frac{R_1}{x-\rho_1}\right) y' + \frac{(s_0 x + r_0)\,(a-x)}{(x-\rho_3)^2\,(x-\rho_2)^2\,(x-\rho_1)^2}\, y \tag{101}$$

This is a Heun equation with its four regular singularites at $\{\rho_1, \rho_2, \rho_3, a\}$, where $R_1 + R_2 + R_3 = -3$,

$$R_2 = \frac{{\rho_2}^2 - (s_1 + \rho_3 + \rho_1)\,\rho_2 + \rho_3\rho_1 - r_1}{(\rho_1 - \rho_2)\,(\rho_2 - \rho_3)} \tag{102}$$

and $\{R_1, R_3\}$ are obtained from R_2 multiplying by -1 and respectively swapping $\rho_2 \leftrightarrow \rho_1$ and $\rho_2 \leftrightarrow \rho_3$. Through the confluence processes which coalesce singularities in (1), generating the CHE and BHE confluent equations, one coalesces the corresponding singularities ρ_i of (101), generating the same type of confluent equations, and that is equivalent to having multiple roots ρ_i in the AIR (11).

By rewriting (101) in normal form (see (58)), the number of irreducible parameters involved is shown to be four instead of six as in (2). That explains the restrictions on the Heun parameters of the BHE, CHE and GHE families discussed in the previous sections. In the three cases, one parameter is fixed and another one is dependent on those remaining.

Different from the BHE, CHE and GHE cases, in the case of the DHE (5) and THE (6) the approach considered in this paper does not lead to new solutions. That can be seen by applying to (101) the DHE and THE confluence processes [2], in both cases arriving at equations already of ${}_p\mathrm{F}_q$ type. That status of things is somewhat expected: the AIR class is generated from the three canonical forms (14) and these are, in their general form, already linked to GHE, CHE and BHE families.

Independent of the possibility, developed here, of expressing the solutions to the BHE (20), CHE (41) and GHE (58) normal forms *without* introducing "Heun functions", these functions have been developed consistently during the last years and will most certainly form part of the standard mathematical language

[19] For that purpose, apply first $\{x \leftrightarrow y\}$ to (11) at $\{s_2 = 1, r_2 = -a\}$, then apply (12) to the resulting Riccati equation.

in the near future. That can be inferred from the relevance of Heun equations in applications. In this framework, the results of this paper could be seen as the identification of multi-parameter special cases of Heun functions of the BHE, CHE and GHE types, respectively admitting the integral representations (36), (51) and (64), and the linear combinations of ${}_p\mathrm{F}_q$ functions with variable coefficients (85), (89) and (92). The mathematical properties and the relevance of these special cases in applications require further investigation.

This link between Heun and ${}_p\mathrm{F}_q$ second order linear equations through Abel non-linear equations of first order seems to be the simplest case of a link between linear equations with N and N-1 singularities, through "Abel AIR like" equations, for which the numerator of the right-hand-side has degree N-1 at most[20]. For example, if instead of (11) we depart from

$$y' = \frac{(y-\rho_1)\,(y-\rho_2)\,(y-\rho_3)\,(y-\rho_4)}{(s_2\,x^2+s_1\,x+s_0)\,y+r_2\,x^2+r_1\,x+r_0} \tag{103}$$

that is, an equation with structure similar to the AIR (11) but whose numerator of the right-hand-side is of degree four, then by applying $\{x \leftrightarrow y\}$ to obtain a Riccati equation and transforming the latter into a second order linear equation, we obtain an equation similar to (101) but with five regular singular points,

$$y'' = \left(\frac{1}{x-a}+\frac{R_4}{x-\rho_4}+\frac{R_3}{x-\rho_3}+\frac{R_2}{x-\rho_2}+\frac{R_1}{x-\rho_1}\right)y' + \frac{(s_0 x+r_0)\,(a-x)}{(x-\rho_4)^2\,(x-\rho_3)^2\,(x-\rho_2)^2\,(x-\rho_1)^2}\,y, \tag{104}$$

which, together with its confluent cases, can be linked through (103), this time to the Heun equations, using the same approach presented in the previous sections relating Heun to ${}_p\mathrm{F}_q$ equations.

Analogously, n^{th} order $(n>2)$ linear equations in $y(x)$ can also be reduced to "Riccati like" non-linear equations of order $n-1$, due to their invariance under scalings of y. It is therefore reasonable to expect that a link equivalent to the one discussed in this work also exists between linear equations with N and N-1 singularities in the n^{th} $(n>2)$ order case.

Appendix

In sec. 2, 3 and 4, solutions were derived for the equations in *normal form* BHE (20), CHE (41) and GHE (58). Given a second order linear ODE

$$y'' + c_1 y' + c_0 y = 0 \tag{105}$$

where the $c_i \equiv c_i(x)$, the corresponding normal form,

$$y'' + \left(c_0 - ({c_1}^2 + 2\,{c_1}')/4\right) y = 0 \tag{106}$$

is obtained by changing $y \rightarrow \exp(-\int c_1\,dx/2)\,y$.

Regarding Heun equations, one advantage of the normal form is that the general or confluent type of the equation is evident in the partial fraction decomposition of the coefficient of y (see eqs. (2) to (6)). Also, two different equations related by $y \rightarrow P(x)\,y$ have the same normal form and recognizing this equivalence is relevant for computational purposes. On the other hand, for different reasons, special functions are frequently defined as solutions to equations in *canonical form*. This appendix relates the normal and canonical forms of the BHE, CHE and GHE, expressed in terms of irreducible parameters[21], following the notation of [2], thus permitting a simple translation of the results presented.

[20] For Abel equations of the first kind, $N=3$; for Abel equations of the second kind like (9), $N \leq 3$.

[21] A few of the equations shown in the paper are repeated here for ease of reading.

The Biconfluent Heun equation

The BHE canonical form is given in terms of four constant parameters $\{\alpha, \beta, \gamma, \delta\}$ by

$$y'' + \left(\frac{1+\alpha}{x} - \beta - 2\,x\right) y' + \left(\gamma - \alpha - 2 - \frac{\delta + (1+\alpha)\,\beta}{2\,x}\right) y = 0; \tag{107}$$

The BHE in normal form (4) in terms of four parameters $\{B, C, D, E\}$ is

$$y'' + \left(-x^2 + B\,x + C + \frac{D}{x} + \frac{E}{x^2}\right) y = 0 \tag{108}$$

The BHE normal form (20) solved in sec. 2, there written in terms of two parameters $\{\sigma, \tau\}$, is

$$y'' - \left(x^2 + 2\,\sigma\,x + \tau^2 + \frac{\tau}{x} + \frac{3}{4\,x^2}\right) y = 0 \tag{109}$$

The parameters $\{B, C, D, E\}$ in (108) are related to $\{\sigma, \tau\}$ by

$$B = -2\,\sigma, \quad C = -D^2, \quad D = -\tau, \quad E = -3/4 \tag{110}$$

The parameters $\{\alpha, \beta, \gamma, \delta\}$ in (107) are related to $\{B, C, D, E\}$ by

$$\alpha^2 = -4\,E + 1, \quad \beta = -B, \quad \gamma = B^2/4 + C, \quad \delta = -2\,D \tag{111}$$

So the parameters $\{\alpha, \beta, \gamma, \delta\}$ are related to $\{\sigma, \tau\}$ by

$$\alpha^2 = 4, \quad \beta = 2\,\sigma, \quad \gamma = \sigma^2 - \tau^2, \quad \delta = 2\,\tau \tag{112}$$

At these values of $\{\alpha, \beta, \gamma, \delta\}$, for $\sigma = \pm\tau$, (107) admits Liovillian solutions and for $\sigma^2 \neq \tau^2$ the solution is obtained from (85).

The Confluent Heun equation

The CHE canonical form is given in terms of five constant parameters $\{\alpha, \beta, \gamma, \delta, \eta\}$ by

$$y'' + \left(\alpha + \frac{\beta+1}{x} + \frac{\gamma-1}{x-1}\right) y' + \frac{(2\,\delta + \alpha\,(\beta+\gamma+2))\,x + 2\,\eta + \beta + (\gamma-\alpha)\,(\beta+1)}{2\,x\,(x-1)}\, y = 0 \tag{113}$$

The CHE in normal form (3) in terms of five parameters $\{A, B, C, D, E\}$ is

$$y'' + \left(A + \frac{B}{x} + \frac{C}{x-1} + \frac{D}{x^2} + \frac{E}{(x-1)^2}\right) y = 0 \tag{114}$$

The CHE normal form (41) solved in sec. 3, there written in terms of three parameters $\{\lambda, \sigma, \tau\}$, is

$$y'' - \left(\lambda^2 + \frac{2\,(\sigma-1)\lambda^2 - \tau\,\lambda + 1/2}{x} + \frac{\tau\lambda - 1/2}{x-1} + \frac{(\tau^2 - 2\,\sigma + 1)\,\lambda^2 - 1/4}{x^2} + \frac{3}{4\,(x-1)^2}\right) y = 0 \tag{115}$$

The parameters $\{A, B, C, D, E\}$ in (114) are related to $\{\lambda, \sigma, \tau\}$ by

$$A = -\lambda^2, \quad B = 2\,(1-\sigma)\,\lambda^2 + \tau\,\lambda - \frac{1}{2}, \quad C = \frac{1}{2} - \tau\,\lambda, \quad D = \frac{1}{4} + \left(2\,\sigma - \tau^2 - 1\right)\lambda^2, \quad E = -3/4, \tag{116}$$

The parameters $\{\alpha, \beta, \gamma, \delta, \eta\}$ in (113) are related to $\{A, B, C, D, E\}$ by

$$\alpha^2 = -4\,A, \quad \beta^2 = -4\,D + 1, \quad \gamma^2 = 4\,\gamma - 4\,E - 3 \quad \delta = C + B - \alpha, \quad \eta = -\frac{1}{2} - B - \beta \tag{117}$$

So the relation between $\{\alpha, \beta, \gamma, \delta, \eta\}$ and $\{\lambda, \sigma, \tau\}$ is

$$\alpha^2 = 4\,\lambda^2, \quad \beta^2 = 4\left(1 - 2\,\sigma + \tau^2\right)\lambda^2 \quad \gamma^2 = 4\,\gamma, \quad \delta = 2\,(1-\sigma)\,\lambda^2 - \alpha, \quad \eta = 2\,(\sigma - 1)\,\lambda^2 - \tau\,\lambda - \beta \tag{118}$$

At these values of $\{\alpha, \beta, \gamma, \delta, \eta\}$, the CHE (113) admits Liouvillian solutions for $\sigma = \pm\tau$, and for $\sigma^2 \neq \tau^2$ the solution is obtained from (89).

The General Heun equation

The GHE canonical form is written in terms of seven constant parameters $\{\alpha, \beta, \gamma, \delta, \epsilon, a, q\}$ as

$$y'' + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right) y' + \frac{\alpha\,\beta\,x - q}{x\,(x-1)\,(x-a)}\, y = 0 \tag{119}$$

where $\gamma + \delta + \epsilon = \alpha + \beta + 1$ and $a \neq 0, a \neq 1$. In [2], the numerator of the coefficient of y of this equation is written as $\alpha\,\beta\,(x - h)$. The notation in (119) is the one used in [1], was apparently first adopted in [27], and has the advantage that one can take $\alpha = 0$ (or $\beta = 0$) without eliminating the term in y completely[22]. The GHE in normal form (2) in terms of six parameters $\{a, A, B, D, E, F\}$ is

$$y'' + \left(\frac{A}{x} + \frac{B}{x-1} - \frac{A+B}{x-a} + \frac{D}{x^2} + \frac{E}{(x-1)^2} + \frac{F}{(x-a)^2}\right) y = 0 \tag{120}$$

The GHE normal form solved in sec. 4 (see (58)), written in terms of four parameters $\{a, \sigma, \tau, \Delta\}$, is

$$\begin{aligned} y'' = \Bigg(& \frac{2\,a^2\,(a-1)\,\Delta^2 - 2\sigma\,a\,(2\,a-1)\,\Delta + \left(2\,\tau^2 - 1/2\right)a + \tau + 1/2}{a\,x} \\ & - \frac{2\left(a\,(a-1)^2\,\Delta^2 - \sigma\,(2\,a-1)\,(a-1)\,\Delta + (\tau - 1/2)\,((\tau + 1/2)\,a - \tau)\right)}{(a-1)\,(x-1)} + \frac{\tau - a + 1/2}{a\,(a-1)\,(x-a)} \\ & + \frac{a^2\Delta^2 - 2\,a\,\sigma\,\Delta + \tau^2 - 1/4}{x^2} + \frac{(a-1)^2\,\Delta^2 - 2\,\sigma\,(a-1)\,\Delta + \tau^2 - 1/4}{(x-1)^2} + \frac{3}{4\,(x-a)^2}\Bigg)\, y \end{aligned} \tag{121}$$

The parameters $\{A, B, D, E, F\}$ in (120) are related to $\{a, \sigma, \tau, \Delta\}$ by

$$\begin{aligned} A &= -2\,a\,(a-1)\,\Delta^2 + 2\,(2\,a-1)\,\sigma\,\Delta - 2\,\tau^2 - \frac{\tau + 1/2}{a} + \frac{1}{2}, \\ B &= 2\,a\,(a-1)\,\Delta^2 - 2\,(2\,a-1)\,\sigma\,\Delta + 2\,\tau^2 + \frac{\tau - a/2}{a-1}, \\ D &= -a^2\Delta^2 + 2\,a\,\sigma\,\Delta - \tau^2 + 1/4, \\ E &= -(a-1)^2\,\Delta^2 + 2\,(a-1)\,\sigma\,\Delta - \tau^2 + 1/4, \\ F &= -3/4 \end{aligned} \tag{122}$$

The parameters $\{\alpha, \beta, \gamma, \delta, \epsilon, q\}$ in (119) are related to $\{a, A, B, D, E, F\}$ by

[22] The differences in notation with respect to [2] are the coefficient of y and the use of $\{\gamma, \delta, \epsilon, \alpha, \beta, q\}$ in place of $\{\alpha, \beta, \gamma, \delta, \eta, h\}$.

$$
\begin{aligned}
\gamma^2 &= -4\,D + 2\,\gamma, \qquad \delta^2 = -4\,E + 2\,\delta, \qquad \epsilon^2 = -4\,F + 2\,\epsilon,\\
\alpha^2 &= (\delta + \epsilon + \gamma - 1)\,\alpha - \frac{(\gamma+\delta)\,\epsilon}{2} - \frac{\gamma\,\delta}{2} + (a-1)\,B + a\,A\\
\beta &= \gamma + \delta + \epsilon - \alpha - 1, \qquad q = \frac{(a\,\delta + \epsilon)\,\gamma}{2} - a\,A
\end{aligned}
$$

So the parameters $\{\alpha, \beta, \gamma, \delta, \epsilon, q\}$ are related to $\{a, \sigma, \tau, \Delta\}$ by

$$
\begin{aligned}
\gamma^2 &= 2\,(2\,a^2\Delta^2 - 4\,a\,\sigma\,\Delta + 2\,\tau^2 + \gamma) - 1\\
\delta^2 &= 2\,(2\,(a-1)^2\,\Delta^2 + 4\,\sigma\,(1-a)\,\Delta + \delta + 2\,\tau^2) - 1\\
\epsilon^2 &= 2\,\epsilon + 3\\
\alpha^2 &= 2\,a\,(1-a)\,\Delta^2 + 2\,(2\,a-1)\,\sigma\,\Delta + (\gamma+\delta+\epsilon-1)\,\alpha - 2\,\tau^2 - (\gamma\,\delta + 1 + (\gamma+\delta)\,\epsilon)/2\\
\beta &= \gamma + \delta + \epsilon - \alpha - 1\\
q &= 2\,a\,\Delta\,(a\,(a-1)\,\Delta - \sigma\,(2\,a-1)) + \frac{a\left(\gamma\,\delta + 4\,\tau^2 - 1\right)}{2} + \frac{\gamma\,\epsilon}{2} + \tau + \frac{1}{2}
\end{aligned}
\tag{123}
$$

At these values of $\{\alpha, \beta, \gamma, \delta, \epsilon, q\}$, for $\sigma = \pm\tau$ the GHE (119) admits Liouvillian solutions, and for $\sigma^2 \neq \tau^2$ the solution is obtained from (92).

Verifying solutions using symbolic computation

In presentations like this one, where equations and solutions involving many parameters and non trivial special functions are involved, it is of use to be able to verify the correctness of the solutions derived, in some way alternative to the one presented. For that purpose, the input lines, written in the Maple symbolic computation syntax, for the BHE (20), the CHE (41) and the GHE (58) equations and their respective solutions (85), (89) and (92), are given, so that they can be copied from the online version of this paper.

The 2-parameter BHE (20) is written in Maple syntax as

```
> BHE := diff(y(x),x,x) - (x^2 + 2*sigma*x + tau^2 + tau/x + 3/4/x^2)*y(x) = 0;
```

and its solution (85) is written as

```
> BHE_sol := y = exp(-sigma*x-1/2*x^2)/x^(1/2)/(x+sigma)*((Lambda*KummerU(1/4*tau^2
> - 1/4*sigma^2,1/2,(x+sigma)^2)-4*KummerU(1/4*tau^2-1/4*sigma^2-1,1/2,(x+sigma)^2))
> * _C1+((tau^2-sigma^2-2)*KummerM(1/4*tau^2-1/4*sigma^2-1,1/2,(x+sigma)^2)-Lambda
> * KummerM(1/4*tau^2-1/4*sigma^2,1/2,(x+sigma)^2))*_C2);
> Lambda := sigma^2+tau^2+4*x^2+2*(3*sigma-tau)*x-2*sigma*tau-2;
```

After entering these lines in a Maple session, to verify this solution one can use the Maple **odetest** command, as in `> odetest( BHE_sol, BHE );` which returns zero, confirming that the solution cancels the equation. The 3-parameter CHE (41) is written in Maple syntax as

```
> CHE := diff(y(x),x,x) = ((-1+2*tau*lambda)/(2*x-2)+1/2*(1+(-4+4*sigma)
> * lambda^2-2*tau*lambda)/x+1/4*(-1+(4-8*sigma+4*tau^2)*lambda^2)/x^2
> + 3/4/(x-1)^2+lambda^2)*y(x);
```

and its solution (89) is written as

```
> CHE_sol := y(x) = 1/(x-1)^(1/2)*(((tau+sigma)*lambda*WhittakerM(mu,nu,2*lambda*x)
> + ((-sigma+1)*lambda-nu)*WhittakerM(-1+mu,nu,2*lambda*x))*_C1+(lambda*(tau-sigma)
> * WhittakerW(-1+mu,nu,2*lambda*x)+WhittakerW(mu,nu,2*lambda*x))*_C2);
> mu := 1/2-lambda*sigma+lambda; nu := lambda*(-2*sigma+1+tau^2)^(1/2);
```

The 4-parameter GHE (58) is written in Maple syntax as

```
> GHE := diff(y(x),x,x) = ((2*a^2*(a-1)*Delta^2-2*sigma*a*(2*a-1)*Delta
> + (2*tau^2-1/2)*a+tau+1/2)/x/a-2*(a*(a-1)^2*Delta^2-sigma*(2*a-1)*(a-1)*Delta
> + (tau-1/2)*((tau+1/2)*a-tau))/(x-1)/(a-1)+(tau-a+1/2)/a/(a-1)/(x-a)
> + (Delta^2*a^2-2*a*sigma*Delta+tau^2-1/4)/x^2+((a-1)^2*Delta^2-2*(a-1)*sigma
> * Delta+tau^2-1/4)/(x-1)^2+3/4/(x-a)^2)*y(x);
```

and its solution (92) is written as

```
> GHE_sol := y(x) = 1/(x-a)^(1/2)*(x-1)^(1/2+Sigma)*((1/2*(Tau-Sigma-Delta)
> * (-Sigma+Delta-1+Tau)*(x^(5/2-Tau)-x^(3/2-Tau))*hypergeom([Sigma-Delta+2
> - Tau, Sigma+Delta-Tau+1],[2-2*Tau],x)+hypergeom([Sigma+Delta-Tau, Sigma-Delta
> + 1-Tau],[1-2*Tau],x)*((Tau-Sigma-Delta)*x^(3/2-Tau)+x^(1/2-Tau)
> * (-Tau+a*Delta+tau))*(-1/2+Tau))*_C1+_C2*(1/2*(Sigma+Delta+Tau)*(Sigma-Delta
> + 1+Tau)*(-x^(5/2+Tau)+x^(3/2+Tau))*hypergeom([Sigma-Delta+2+Tau, Sigma
> + Delta+Tau+1],[2+2*Tau],x)+(1/2+Tau)*((-Tau-Sigma-Delta)*x^(3/2+Tau)
> + x^(1/2+Tau)*(Tau+a*Delta+tau))
> * hypergeom([Sigma+Delta+Tau, Sigma-Delta+1+Tau],[1+2*Tau],x)));
> Sigma := sqrt((a-1)^2*Delta^2-2*(a-1)*sigma*Delta+tau^2);
> Tau := sqrt(a^2*Delta^2-2*a*sigma*Delta+tau^2);
```

Acknowledgments

This work was supported by the MITACS NCE, the Centre of Experimental and Constructive Mathematics of Simon Fraser University and the Maplesoft division of Waterloo Maple Inc. The author thanks K. von Bülow for a careful reading of this paper, as well as one of the referees for her/his fruitful comments.

References

[1] "Heun's Differential Equations", Edited by A. Ronveaux, Oxford University Press (1995).

[2] A. Decarreau, M.Cl. Dumont-Lepage, P.Maroni, A.Robert, A.Ronveaux, "Formes Canoniques de Équations confluentes de l'équation de Heun", Annales de la Société Scientifique de Bruxelles, T. 92, I-II, pp. 53-78 (1978).

[3] A.O. Starinets, "Quasenormal modes of near extremal black branes", Physical Review D **66**, 124013 (2002).

[4] O.I. Tolstikhin, M. Matsuzawa, "Hyperspherical elliptic harmonics and their relation to the Heun equation", Physical Review A **63**, 032510 (2001).

[5] L. Anguelova, P. Langfelder, "Massive gravitino propagator in maximmaly symmetric spaces and fermions in dS/CFT", Journal of High Energy Physics 057 (2003).

[6] F. Finkel, A. González-López, A. L. Maroto, and Miguel Á. Rodríguez, "The Lamé equation in parametric resonance after inflation", Physical Review D **62** 103515 (1-7) (2000).

[7] S.Y. Slavyanov and W.Lay. "Special Functions, A Unified Theory Based on Singularities", Oxford Mathematical Monographs (2000).

[8] P.A. Clarkson, P.J. Olver, "Symmetry and Chazy Equation", Journal of Differential Equations 124, 225-246 (1996).

[9] A. Ronveaux, "Factorization of the Heun's differential operator", Applied Mathematics and Computation 141, 177-184 (2003).

[10] A. Ishkhanyan1 and K.A. Suominen, “New solutions of Heun’s general equation”, J. Phys. A: Math. Gen. 36, L81-L85, (2003).

[11] R.S. Maier, “Transforming the Heun equation to the Hypergeometric equation: 1. Polynomial Transformations”, Foundations of Computational Mathematics ’02, (2002).

[12] A.O. Smirnov, “Elliptic solitons and Heun’s equation”, CRM Proceedings and Lecture Notes v.32, 287-305 (2002).

[13] H. Exton, “A new solution of the biconfluent Heun equation”, Rendiconti di Mathematica Serie VII, Volume 18, 615-622 (1998).

[14] A. Ishkhanyan1 and K.A. Suominen, “Analytic treatment of the polariton problem for a smooth interface”, J. Phys. A: Math. Gen. 34, L591-L598, (2001).

[15] A.V. Shanin and R.V. Craster, “Removing false singular points as a method of solving ordinary differential equations”, European Journal of Applied Mathematics, Vol. 13, 617-639 (2002).

[16] E.S. Cheb-Terrab, A.D. Roche, “An Abel ODE class generalizing known integrable classes”, European Journal of Applied Mathematics, Vol. 14, No. 2, 217-229 (2003).

[17] E.S. Cheb-Terrab, “A connection between Abel and ${}_p\mathrm{F}_q$ hypergeometric differential equations”, submitted to the European Journal of Applied Mathematics (http://arxiv.org/math-ph/0402040).

[18] L.Chan, E.S. Cheb-Terrab, “Non Liouvillian solutions for second order linear ODEs”, Proceedings of ISSAC’04, Santander, Spain (2004).

[19] L. Gagnon, P. Winterniz, “Non-Painlevé reductions of nonlinear Schrödinger equations”, Physical Review A **42** 8 (1990).

[20] E. Kamke, “Differentialgleichungen”, N.Y. Chelsea Publ. Co. (1947).

[21] A.D. Polyanin, V.F. Zaitsev, “Handbook of Exact Solutions for Ordinary Differential Equations”. CRC Press, Boca Raton (1995).

[22] P.L. Sachdev, “A Compendium of Nonlinear Ordinary Differential Equations”. John Wiley & Sons (1997).

[23] E.S. Cheb-Terrab, A.D. Roche, “Abel Equations: Equivalence and Integrable Classes”, Computer Physics Communications 130 (2000).

[24] J.Kovacic, “An algorithm for solving second order linear homogeneous equations”, J. Symb. Comp. 2, p. 3-43 (1986).

[25] M. Abramowitz and I. A. Stegun, “Handbook of mathematical functions”, Dover (1964).

[26] H.Stephani, Differential equations: their solution using symmetries, Cambridge University Press, (1989).

[27] Erdélyi et al, “Higher Transcendental Functions”, Vol III, The Bateman Manuscript Project, McGraw-Hill (1955).

[28] E.T. Whittaker and G.N. Watson, “A Course in Modern Analysis”, 4th ed. Cambridge University Press (1990).